It is very simple now to demonstrate that existence of variety of various electric and magnetic fields are possible even in vacuum, i.e. without charges or current.

$\rho = 0 \quad \vec{j} = 0$

That means Maxwell's equations can be simplified to:

$$\begin{cases} \operatorname{div} \vec{E} = 0 \\ \operatorname{div} \vec{H} = 0 \\ \operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad * \\ \operatorname{curl} \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad ** \end{cases}$$

In order to see that this system has solution (i.e. waves) we will play with these two equations in order to solve it towards $\vec{E}$ only:

$$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad *+$$

$$\Downarrow$$

$$\operatorname{curl}(\operatorname{curl} \vec{E}) = -\operatorname{curl}\left(\frac{\partial \vec{B}}{\partial t}\right)$$

Little mathematical interlude

$$\boxed{\vec{\nabla} \times [\vec{\nabla} \times \vec{A}] = \vec{\nabla}(\vec{\nabla}\vec{A}) - \vec{\nabla}^2 \vec{A}} \quad \text{Homework}$$

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -\left[\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}\right]$$

$$\vec{\nabla}(\vec{\nabla}\vec{E}) - \vec{\nabla}^2 \vec{E} = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

$$(\vec{\nabla}\vec{E}) = \operatorname{div} \vec{E} = 0$$

$$\vec{\nabla}^2 \vec{E} = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

At the same time:

$$* \; c^2 [\vec{\nabla} \times \vec{B}] = \frac{\partial \vec{E}}{\partial t} \Rightarrow -\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \Rightarrow \text{wave equation for the field } \vec{E}$$

Let's try to derive it for field $\vec{B}$:

$$c^2 \operatorname{rot} \vec{B} = \frac{\partial \vec{E}}{\partial t}$$

$$c[\vec{\nabla} \times [\vec{\nabla} \times \vec{B}]] = \frac{\partial}{\partial t}[\vec{\nabla} \times \vec{E}]$$

$$\vec{\nabla}^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \Rightarrow \text{wave equation for field } \vec{B}$$

$$\begin{cases} \vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \vec{\nabla}^2 \vec{B} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \end{cases}$$

The solutions to these wave equations are running waves, that propagate with speed c.

If we would look only one coordinate c (that means it is plane wave):

$$\begin{cases} \frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \\ \frac{\partial^2 \vec{B}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \end{cases}$$

The solution of these equations will be any function of argument $(t - \frac{z}{c})$

$$\vec{E}(z,t) = E\left(t - \frac{z}{c}\right)$$

Homework: Prove it

"−" means that wave propagates along z

"+" means that wave propagates against z.

Running plane harmonic waves

The wave that is described with cosine harmonic function is a plane harmonic (monochromatic) wave:

$$\vec{E}(z,t) = A\cos\left(t - \frac{z}{c}\right)$$

Why the wave called plane?

E is the same in the z=const plane.

A - amplitude

ω - frequency.

$$E(z,t) = A\cos\left(\omega t - \frac{\omega z}{c}\right)$$

$\frac{\omega}{c} = k$ k - wave number

$k = \frac{\omega}{c} = \frac{2\pi}{\lambda}$

In vector form:

$$E(\vec{r},t) = A\cos(\omega t - \vec{k}\vec{r})$$

$\vec{k}$ - wave vector, i.e. vector of direction of field propagation

Phase of the wave and phase speed

The argument $(\omega t - \vec{k}\vec{r})$ is called the phase of the wave. In this case, the equation:

$$(\omega t - \vec{k}\vec{r}) = \text{const}$$

is the wave surface

What would be the speed?

$$(\omega t - \vec{k}\vec{r}_1) = (\omega t_2 - \vec{k}\vec{r}_2)$$

$$(\omega t - \vec{k}\vec{r}) = \text{const} \Big|_{\frac{d}{dt}}$$

$$\frac{d\vec{r}}{dt} = \frac{\omega}{k}$$

The surface is moving in space with speed $v = \frac{\omega}{k}$ – phase speed. It is obvious that $v = c$ in vacuum.

Let's express harmonic function through general complex function.

Spherical waves

These are the waves where

$$\vec{E}(\vec{r},t) \quad r\vec{E} = r\vec{E}(r,t)$$

These waves are of major importance to optics and electromagnetism as they describe emission of point source of light.

Let's find solution:

$$\operatorname{div} \vec{E} = \Delta \vec{E}$$

$$\Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \quad (1)$$

The Laplacian in case of spherical wave:

$$\Delta \vec{E} = \frac{1}{r} \frac{\partial^2}{\partial r^2}(r\vec{E}) \quad (2)$$

Then we have from (1) and (2):

$$\frac{1}{r} \frac{\partial^2}{\partial r^2}(r\vec{E}) = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Downarrow \qquad E = r\vec{E}$$

$$\begin{cases} \frac{\partial^2 E}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} \\ E = E_1\left(t - \frac{r}{c}\right) + E_2\left(t + \frac{r}{c}\right) \end{cases}$$

$$E = \frac{E_1\left(t - \frac{r}{c}\right)}{r} + \frac{E_2\left(t + \frac{r}{c}\right)}{r} \rightarrow \text{spherical}$$

Gaussian beam

In real life it is impossible to have waves as point source of light are not really exist. This leads to spatial variation of waves, i.e. wave amplitude and phase are dependent on coordinate (X,y).

Why Gaussian beams are of particular interest?

1) Coherent radiation with Gauss profile has best directionality that is available due to wave nature of propagation.

2) It is worst about approximation for diffraction limit for space limited beam.

Let's start directionality argument:

We need to solve wave equation, assuming Gauss beam and boundary conditions:

$$\Delta E_x - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

boundary conditions: $E_x = A_0 e^{-\frac{r^2}{\sigma^2}} \cos(\omega t)$

$r = x^2 + y^2$ - distance

The solution will take form:

$$E_x = \frac{1}{2} A_c(r,z) e^{i(\omega t - kz)} + c.c. \quad (1)$$

Here: c.c. - conjugated term

$$A_c = \frac{A_0}{(1 - i\frac{z}{z_d})} e^{\frac{r^2}{(-i\sigma_0^2 + z_d)}} \quad (2)$$

$$z_d = \frac{k\sigma_0^2}{2}$$

We can re-write (2):

$$A_c = A e^{-i\varphi}$$

where A and φ are real amplitude and phase of the wave:

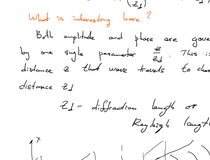
$$A(r,z) = \frac{A_0}{\sqrt{1 + \left(\frac{z}{z_d}\right)^2}} e^{-\frac{r^2}{(1 + \frac{z^2}{z_d^2})\sigma^2}}$$

$$\varphi = \frac{r^2}{\sigma^2} \frac{\frac{z}{z_d}}{1 + \left(\frac{z}{z_d}\right)^2} - \operatorname{atg}\left(\frac{z}{z_d}\right)$$

What is interesting here?

Both amplitude and phase are governed by one single parameter $\frac{z}{z_d}$. This is distance z that wave travels to characteristic distance z_d

z_d - diffraction length or Rayleigh length



When $z \ll z_d$, then

$$E_x = A_0 e^{-\frac{r^2}{\sigma^2}} \cos(\omega t - kz) \rightarrow \text{Gaussian beam}$$

When $z \gg z_d$

$$A(r,z) = A_0 \frac{z_d}{z} \sim \frac{1}{z}$$

and

$$\varphi \sim \frac{r^2}{z_d^2} \rightarrow \text{spherical front}$$